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Computation of the flow between two rotating coaxial disks

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A numerical investigation of the problem of rotating disks is made using the Newton–Raphson method. It is shown that the governing equations may exhibit one, three or five solutions. A physical interpretation of the calculated profiles will be presented. The results computed reveal that both Batchelor and Stewartson analysis yields for high Reynolds numbers results which are in agreement with our observations, i.e. the fluid may rotate as a rigid body or the main body of the fluid may be almost at rest, respectively. Occurrence of a two-cell situation at particular branches will be discussed.

1. Introduction

The study of the steady flow of an incompressible viscous fluid between two infinite rotating disks is of considerable importance since it offers the possibility of calculating an exact solution to the Navier–Stokes equations. In 1921 von Kármán showed that the Navier–Stokes equations describing the flow for a single infinite rotating disk can be reduced by making use of the similarity equations to a set of nonlinear ordinary differential equations. Using this approach he was able to calculate a solution for the flow in the vicinity of an infinite rotating disk. Later Batchelor (1951) and Stewartson (1953) pointed out that the same transformations can be applied to the problem of steady flow between two infinite rotating disks. For infinite Reynolds number they found unique limiting flows which are, however, qualitatively different. Batchelor pointed out that for high Reynolds numbers the flow between the two disks is characterized by the fact that the main body of the fluid rotates with constant angular velocity and that boundary layers develop on both disks. On the other hand, Stewartson predicted profiles for which the velocity outside the boundary layers has only an axial component. Mellor, Chapple & Stokes (1968) and Roberts & Shipman (1976) produced several classes of solutions which are referred to as the multiple-cell solutions. According to Mellor *et al.* a cell is defined as a region bounded by planes parallel to the disks on which the axial velocity vanishes. Mellor *et al.* discovered two one-cell branches, one two-cell branch and one three-cell branch of solutions while recently Roberts & Shipman claimed to have produced solutions with as many as five cells. Nguyen, Ribault & Florent (1975), using the Newton–Raphson method, calculated two solutions for large values of the Reynolds number. Unfortunately, so far the connexions between multiple solutions on the one hand and multiple-cell solutions on the other are not evident.

The goals of this paper are (i) to propose a reliable method which makes it possible to

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calculate the problem for any arbitrary value of the Reynolds number, (ii) to make an attempt to evaluate the number of possible solutions of the governing equations and (iii) to find a relation between multiplicity of solutions and multiple-cell regions.

2. Governing equations

Consider the physical situation of two rotating disks. Cylindrical polar co-ordinates (r, θ, z) are used and the lower disk, in the plane $z = 0$, has an angular velocity Ω while the upper disk, at $z = d$, has an angular velocity $s\Omega$. For steady incompressible flow of a viscous fluid the governing equations are the Navier–Stokes equations, which constitute a set of nonlinear partial differential equations. In this paper the transformations

$$u = r\Omega F(\xi), \quad v = r\Omega G(\xi), \quad w = (\nu\Omega)^{\frac{1}{2}} H(\xi) \tag{2.1}$$

are used to reduce the Navier–Stokes equations to a set of ordinary nonlinear differential equations. Here the velocity components of the fluid are (u, v, w) in the directions of increasing (r, θ, z) respectively, ν is the kinematic viscosity of the fluid and ξ is a new dimensionless axial co-ordinate: $\xi = z/d$.

On using these substitutions, the Navier–Stokes equations result in a set of ordinary nonlinear differential equations:

$$F'' = R^{\frac{1}{2}}HF' + R(F^2 - G^2 + k), \tag{2.2}$$

$$G'' = 2RFG + R^{\frac{1}{2}}G'H, \tag{2.3}$$

$$H' = -2R^{\frac{1}{2}}F, \tag{2.4}$$

where $R = \Omega d^2/\nu$ is the Reynolds number and k is an unknown constant which arises from the pressure equation. From the no-slip condition at both disks, the boundary conditions are

$$F(0) = H(0) = 0, \quad G(0) = 1, \tag{2.5}$$

$$F(1) = H(1) = 0, \quad G(1) = s. \tag{2.6}$$

The problem is now to calculate the functions $F(\xi)$, $G(\xi)$ and $H(\xi)$ for various values of R and s in the ranges $0 \leqslant R < \infty$ and $-1 \leqslant s \leqslant 1$. Notice that six boundary conditions have been formulated for a fifth-order system since the constant k is unknown.

3. Numerical solution of governing equations

The above problem, for higher values of the Reynolds number, is considered in the literature as a particularly difficult numerical problem. So far a number of numerical techniques have been proposed to solve it: Lance & Rogers (1962) and Osborne (1969) used the shooting method, Well (1972) adopted the quasi-linearization procedure of Bellman & Kalaba (1965), Greenspan (1972) made use of the relaxation procedure while Pearson (1965) took advantage of the false transient method. Recently, Nguyen *et al.* (1975) have applied the Newton–Raphson approach, Kubíček, Holodniok & Hlaváček (1976, 1977) differentiation with respect to an actual parameter and one-parameter imbedding, respectively, and Roberts & Shipman (1976) a continuation method for sensitive problems.

To solve this two-point boundary-value problem by means of the shooting method we need to know $F'(0)$, $G'(0)$ and k. Furthermore, to get a solution to our problem, we have to use certain definite values of these missing conditions in order that

$$F(1) = H(1) = 0 \quad \text{and} \quad G(1) = s.$$

Lance & Rogers have shown that the classical shooting method may be used for Reynolds numbers as high as approximately 700–800. For higher values, unfortunately, this procedure fails. This observation is in agreement with our experience. Recently Roberts & Shipman have managed to combine the shooting method with continuation. This method is capable of integrating sensitive two-point boundary-value problems. Indeed, for the case of rotating disks they were able to calculate the solution up to $R \sim 5000$.

The quasi-linearization method, which uses the superposition principle, suffers from the same shortcoming as the classical shooting method. Moreover, the relaxation technique is also unable to calculate easily all solutions for higher values of the Reynolds number (Nguyen *et al.*). The most promising approaches are those where the governing equations are solved simultaneously, e.g. the false transient method (solved by an implicit finite-difference scheme) and the Newton–Raphson method. Both approaches have made it possible to calculate the solution for higher values of the Reynolds number: Pearson (1965) reported a solution for $R = 1000$ whilst Nguyen *et al.* presented results for $R = 7000$.

In this paper we employ the Newton–Raphson method for solution of nonlinear finite-difference equations in a form which can handle the problem of rotating disks in a very convenient way. After the first and second derivatives in (2.2)–(2.4) have been replaced by the finite-difference formulae

$$F'' \approx (F_{i-1} - 2F_i + F_{i+1})/h^2, \quad F' \sim (F_{i+1} - F_{i-1})/2h, \tag{3.1}$$

a set of nonlinear finite-difference equations results:

$$\frac{F_{i-1} - 2F_i + F_{i+1}}{h^2} - R^{\frac{1}{2}} H_i \frac{F_{i+1} - F_{i-1}}{2h} - R(F_i^2 - G_i^2 + k) = 0, \quad i = 2, \ldots, n, \tag{3.2}$$

$$\frac{G_{i-1} - 2G_i + G_{i+1}}{h^2} - 2RF_i G_i - R^{\frac{1}{2}} \frac{G_{i+1} - G_{i-1}}{2h} H_i = 0, \quad i = 2, \ldots, n, \tag{3.3}$$

$$\frac{H_{i+1} - H_{i-1}}{2h} + 2R^{\frac{1}{2}} \frac{F_{i+1} + F_i}{2} = 0, \quad i = 1, \ldots, n. \tag{3.4}$$

Here we have used a uniform mesh with $n+1$ mesh points. To solve this huge set of nonlinear finite-difference equations the Newton–Raphson method has been used. After linearization of (3.2)–(3.4) the set of linear algebraic equations for the new, $(j+1)$th approximation of the variables

$$\mathbf{b}^{\mathrm{T}} = (H_2, F_2, G_2, H_3, F_3, G_3, \ldots, H_n, F_n, G_n, k) \tag{3.5}$$

may be written in the matrix form

$$\mathbf{A}\mathbf{b}^{j+1} = \mathbf{d}. \tag{3.6}$$

Here $\mathbf{A}$ is the Jacobian matrix of the set (3.2)–(3.4) and $\mathbf{d}$ is the vector of the right-hand sides. Both $\mathbf{A}$ and $\mathbf{d}$ are evaluated using $\mathbf{b}^j$. If the governing equations are solved in the order (3.4) for $i = 1$, (3.2) for $i = 2$, (3.3) for $i = 2$, (3.4) for $i = 2$, (3.2) for $i = 3$, ..., (3.4) for $i = n$, then, with respect to the vector of variables (3.5), a seven-diagonal band matrix with a non-zero last column results. From (2.5) and (2.6), $H_1 = 0$, $F_1 = 0$, $G_1 = 1$ and $H_{n+1} = 0$, $G_{n+1} = s$.

Of course, since the matrix $\mathbf{A}$ is sparse efficient elimination algorithms may be used to solve (3.5) (Kubíček 1973). In addition only the diagonal elements (7 diagonals) as well as the last column are stored, so that problems with a high dimension may be solved, i.e. a large number of grid points can be used. The grouping

$$\mathbf{b}^{\mathrm{T}} = (H_2, H_3, \ldots, H_n, F_2, F_3, \ldots, F_n, G_2, G_3, \ldots, G_n, k), \tag{3.7}$$

which was used by Nguyen *et al.*, gives rise to a matrix which does not exhibit the band structure and, as a result, the calculation process is very cumbersome. The algorithm proposed here made it possible to adopt 100 mesh points ($h = 0{\cdot}01$) for the calculations reported. To be sure that this accuracy is sufficient we have recalculated the problem for $R = 625$ with $h = 0{\cdot}005$ and $h = 0{\cdot}0025$, i.e. with 200 and 400 mesh points. All calculations were performed in a double-precision arithmetic, which corresponds to 15 significant digits on the computer Tesla 200. The proposed procedure allowed us to calculate the profiles for very high Reynolds numbers $R \sim 10^5$.

4. Discussion of numerical results

The results reported here have been obtained for the case of disks rotating in the same sense ($s = 0{\cdot}8$). Using the Newton–Raphson technique we have performed a systematic search for $R = 625$ and have found five solutions of the governing equations (2.2)–(2.6). The dependence of k on R has been calculated by continuation of these solutions. For continuation the Newton–Raphson method has again been used; the solution calculated for R has been used as a first guess for $R + \Delta R$. In the vicinity of the branch points of the function $k(R)$ the Newton–Raphson method may fail. However, here a method employing direct evaluation of the branch points can be used (Kubíček & Hlaváček 1975).

The results of our analysis are displayed in figures 1–5; a complete set of tables of the numerical results is available from the authors.

The dependence of the constant k on the Reynolds number is drawn in figure 1, which reveals that a number of branches exist. From this figure it may be inferred that for $R < 205$ only one solution of the governing equations exists. In the region $205 < R < 330$ three solutions to the Navier–Stokes equations have been calculated while for $R > 330$ five solutions are possible. Let us first consider the branch I (see figure 1) in the region of uniqueness. The profiles of the variables F, G and H for $R = 100$ are drawn in figure 2. We can see that the fluid flows from the disk with lower velocity towards the disk with higher velocity, i.e. there is an inflow at the lower disk and an outflow at the upper disk. Beyond a certain axial distance the angular velocity does not change significantly and the fluid rotates with the velocity $G \approx 0{\cdot}9$. In figure 3(a) the profiles of F, G and H are drawn for $R = 275$. For this value of the Reynolds number multiple solutions may exist. As figure 3(a) reveals, the profiles of F, G and H on branch I are very similar to those for $R = 100$ – however the boundary-layer character of the flow is more obvious. It is evident that there is a boundary layer on each disk with fluid moving inwards on the faster disk and outwards over the

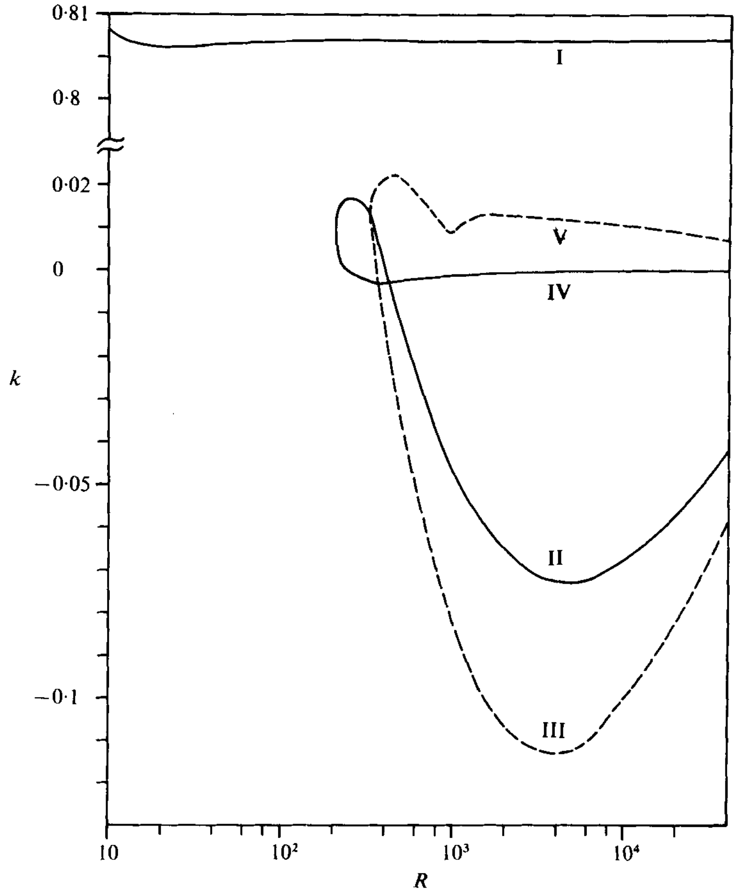


FIGURE 1. Dependence of k on the Reynolds number.

slower disk. The shape of the function G reveals that a substantial part of the fluid is moving *en bloc* in a zone lying between the boundary layers developing near the disks. Flow of this type was discovered in 1951 by Batchelor and hence in this paper branch I will be referred to as the *Batchelor branch.*

However, for $R = 275$ two other solutions of the governing equations exist and are shown in figures 3(*b*) and (*c*). We can see from these figures that both solutions differ from that shown in figure 3(*a*) in all essentials. There is suction on both disks: using the terminology of Mellor *et al.*, a two-cell solution appears. Comparing the profiles displayed in figures 3(*b*) and (*c*) indicates that those on branch II are more asymmetrical. It is worth noting that on branch II the angular velocity does change 'smoothly', and part of the fluid lying in the central zone rotates in the opposite direction.

For $R = 625$ five solutions of (2.2)–(2.6) are possible. The profiles calculated for each particular branch are shown in figures 4(*a–e*). In figure 4(*a*) we may see the Batchelor solution of the Navier–Stokes equations. A substantial part of the fluid rotates as a rigid body. Comparing figures 3(*b*) and 4(*b*) shows that with increasing R all details on branch II remain the same. A similar statement is essentially also true for figures 3(*c*) and 4(*d*). Figure 4(*d*) reveals that the main body of the fluid has no

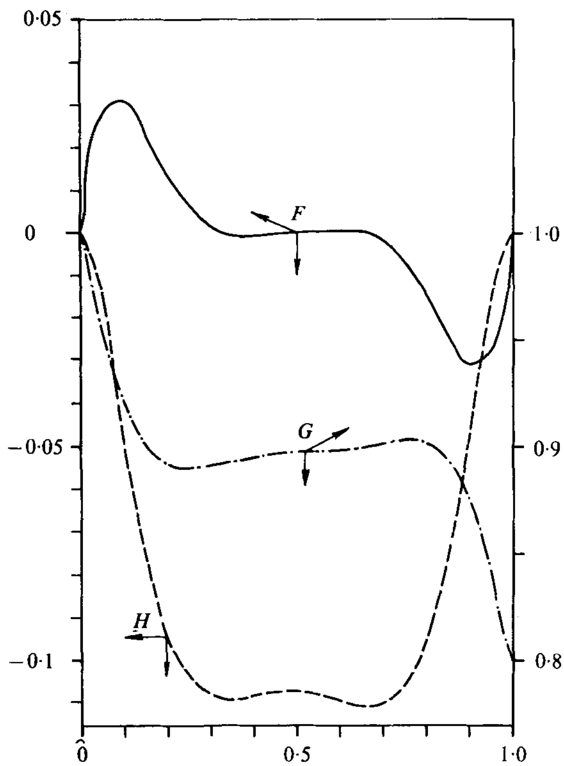


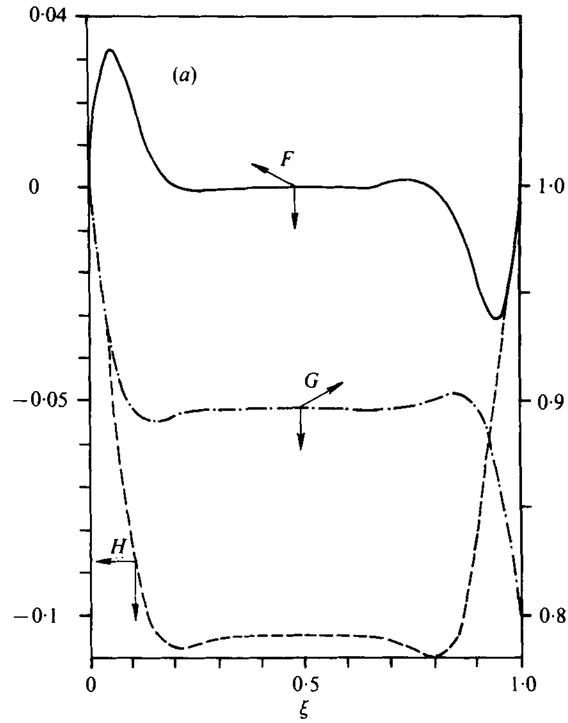
FIGURE 2. Solution for $R = 100$: region of uniqueness. Branch I.

angular velocity. Apparently this is the *Stewartson branch*. There are, however, two additional branches: III and V. On comparing the profiles on branches II and III, it appears that they are mirror-symmetrical. The physical interpretation of the profiles displayed in figure 4 (*e*) is not obvious.

Figures 5 (*a*)–(*e*) show the profiles for $R = 10\,000$. From these figures we may infer that two boundary-layer solutions can exist: the Batchelor branch gives rise to a solution where the fluid rotates as a rigid body and the Stewartson branch to one for which the main body of the fluid does not rotate. In the former case there is suction at the faster disk and blowing at the slower disk while in the latter case there is suction on both disks.

5. Conclusions

From the material presented in this paper it is apparent that more than one solution to the Navier–Stokes equations can exist for the problem of rotating disks. For higher values of the Reynolds number we have found five solutions here. The multiple steady states are the reason why the results in the literature are contradictory. Of course, there is the problem of the stability of particular solutions, which must be solved in the future. A detailed study of the problem for different values of the parameter s is in progress.



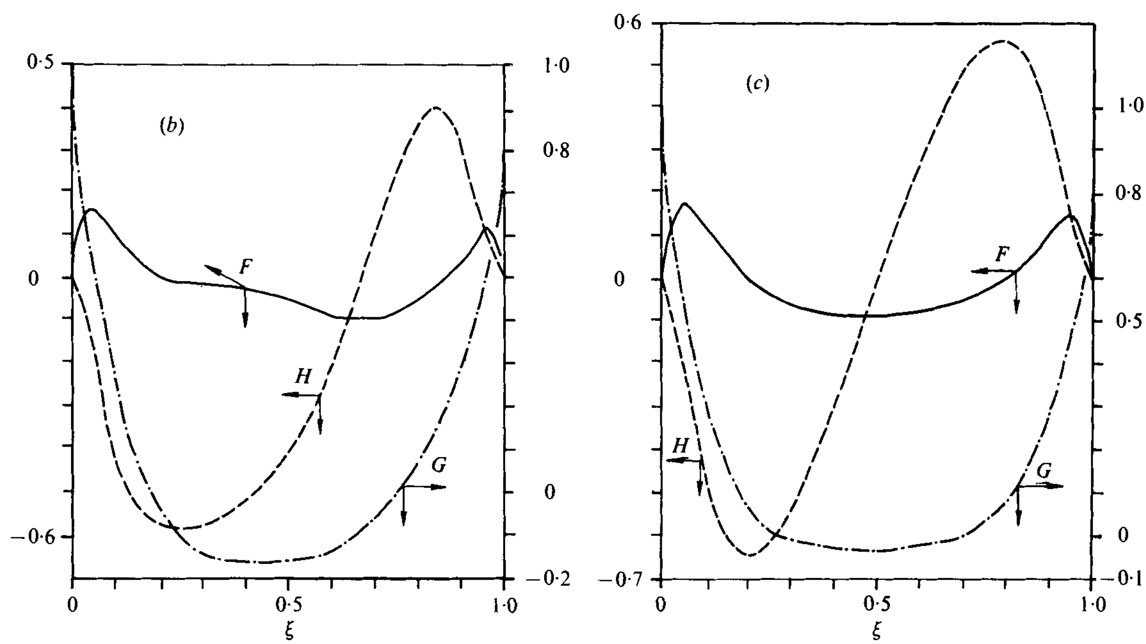
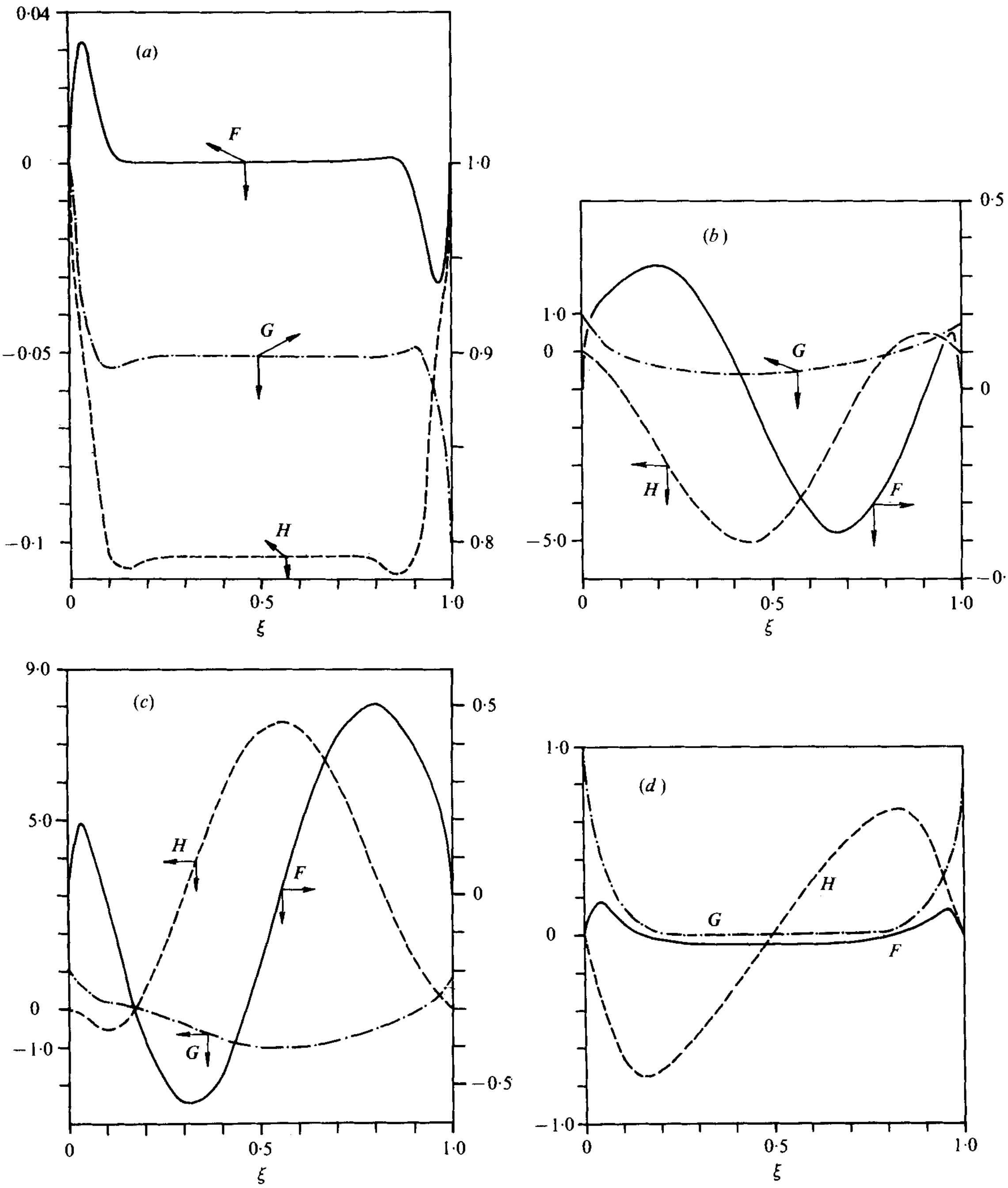


FIGURE 3. Solution for $R = 275$: region of multiplicity (3 solutions possible). (*a*) Branch I. (*b*) Branch II. (*c*) Branch IV.



FIGURES 4 (*a–d*). For legend see facing page.

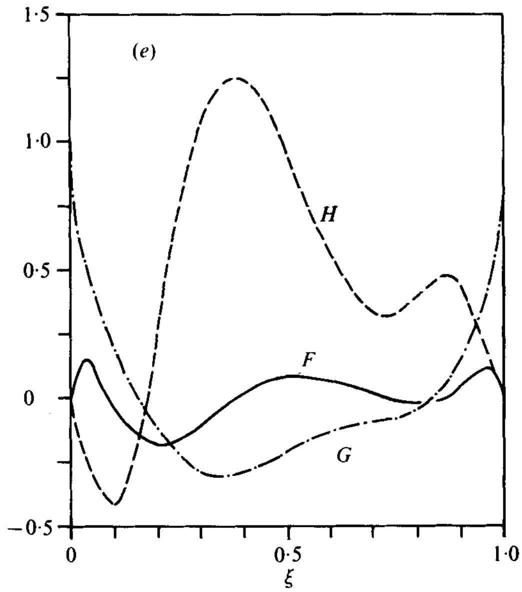
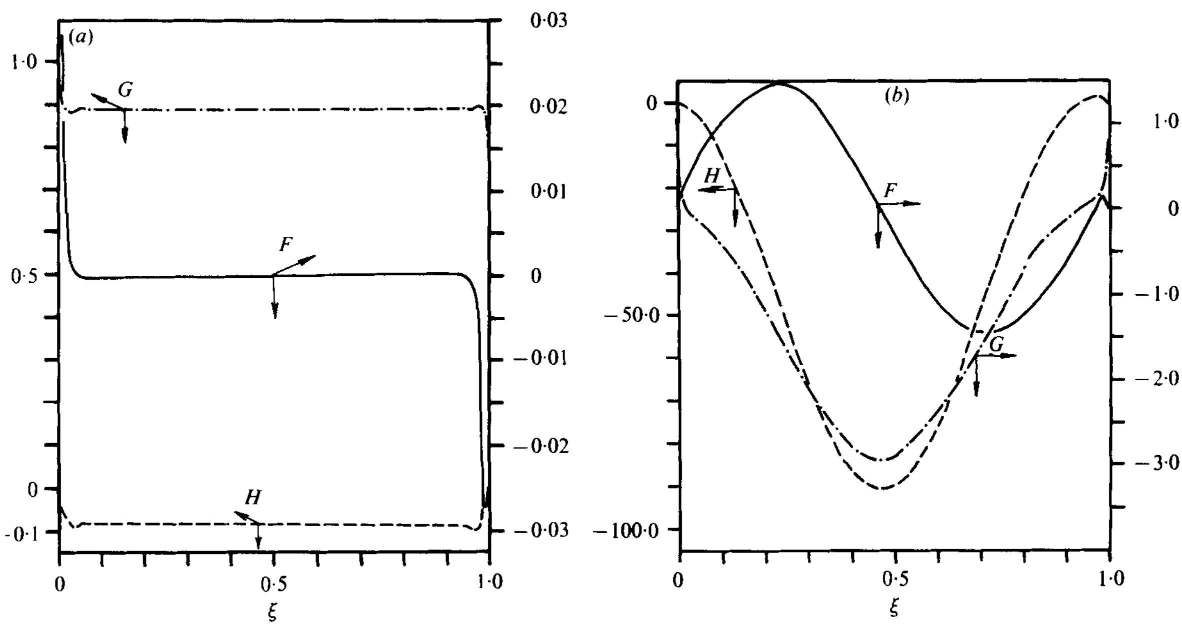


FIGURE 4. Solution for $R = 625$: region of multiplicity (5 solutions possible). (*a*) Branch I. (*b*) Branch II. (*c*) Branch III. (*d*) Branch IV. (*e*) Branch V.



FIGURES 5 (*a*–*b*). For legend see next page.

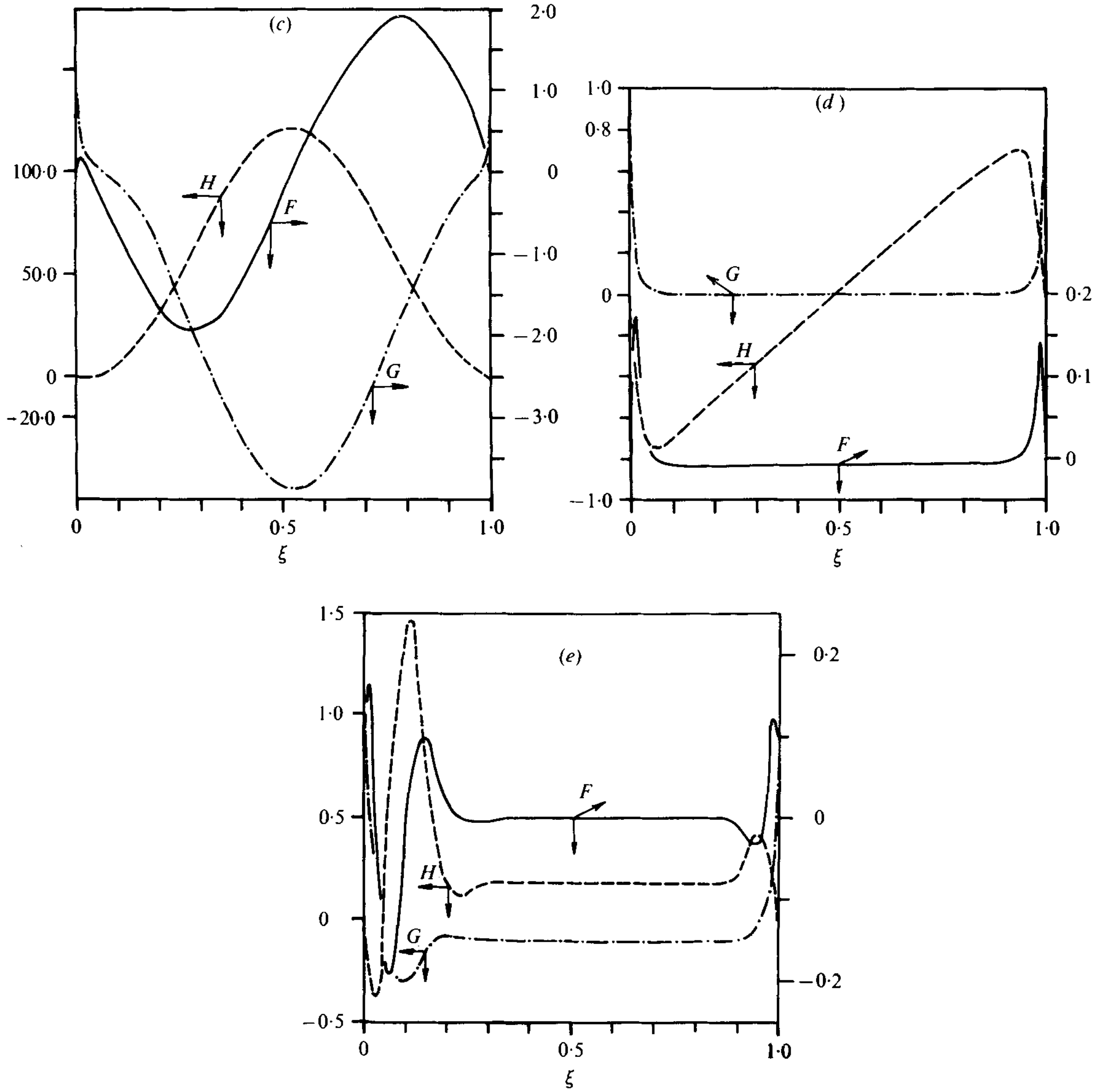


FIGURE 5. Solution for $R = 10\,000$: region of multiplicity (5 solutions possible). (*a*) Branch I. (*b*) Branch II. (*c*) Branch III. (*d*) Branch IV. (*e*) Branch V.

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